The Approximate Networks of Acoustic Filters

By W. P. MASON

The approximate equivalent electrical networks of acoustic filters are developed in this paper, from the lumped-constant approximation networks for electric lines. In terms of this network, design formulæ have been developed for all single band pass filters. It is possible, from these formulae, to determine the physical dimensions of an acoustic filter necessary to have a given attenuation and impedance characteristic.

THE original theory of acoustic filters given by Stewart [1] is based upon the representation of such filters by means of lumped constants in the form of a T network. More recently, the writer [2] has presented a theory of acoustic filters, showing that they are equivalent to a combination of electric lines. Lines, as an approximation, can be represented by networks with lumped constants, and hence an acoustic filter has a lumped-constant approximation network, which should represent the filter well at low frequencies. It is here shown that the network proposed by Stewart is a first approximation to the network of electric lines given in the former paper.[2, 3] This first approximation represents the low pass filter well at low frequencies, but does not very adequately represent the band-pass filters. Accordingly, a second approximation is developed. All of the single band-pass filters have been analyzed and design formulæ are given for them in terms of the second approximation network.

The Approximate Lumped-Constant Networks of Acoustic Filters

An acoustic filter, as developed so far, consists of a main conducting tube, and a side branch. In a symmetrical filter, the side branch is connected to the main conducting tube half-way between the two ends, as shown on Fig. 1. The type of filter obtained depends primarily on

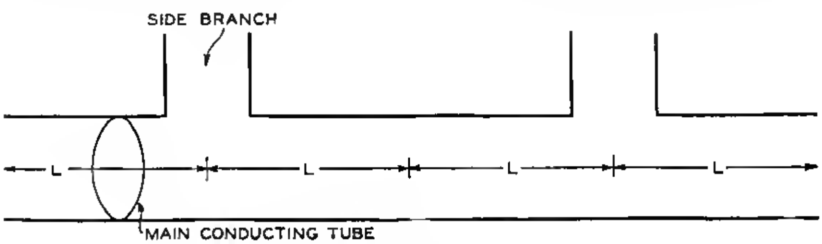


Fig. 1

[1] Stewart, *Phys. Rev.*, 20, pp. 528–551, 1922. *Phys. Rev.*, 25, pp. 90–98, 1925.
[2] Mason, *Bell System Technical Journal*, 6, pp. 258–294, 1927.
[3] This fact has also been pointed out by Stewart, *Journal of the Optical Society*, July 1929, and by Lindsay, *Phys. Rev.*, 25, pp. 652–655, 1929.

what type of side branch is used, the resonances of the latter determining the frequencies of maximum suppression.

The equivalent electrical circuit for an acoustic filter, was shown in a previous paper [2] to be two lines shunted by the impedance of the side branch. This representation is shown on Fig. 2. To obtain a lumped-

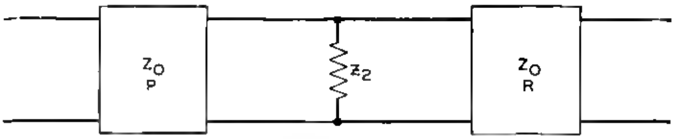


Fig. 2

constant representation for this network, it is necessary first to consider the lumped-constant representation of a line, which is discussed below.

A. Lumped-Constant Representation of a Line

In a previous paper [2] it was shown that the propagation constant of a tube is given by the equation

$$P^2 = \frac{-\omega^2}{c^2}\left[\left(1+\frac{P_0}{S}\sqrt{\frac{\gamma'^2}{2\omega\rho}}\right)-\frac{iP_0}{S}\sqrt{\frac{\gamma'^2}{2\omega\rho}}\right], \tag{1}$$

while the characteristic impedance is given by the expression

$$Z = \frac{\rho c^2 P}{j\omega S}. \tag{2}$$

In these equations ω is 2π times the frequency, c the velocity of sound, P_0 the perimeter of the tube, S its area, ρ the density of the medium and γ'^2, a constant related to the viscosity, which for air has the value 4.25×10^{-4} in c.g.s. units.

A tube is the analogue of an electric line with distributed resistance, inductance, and capacity. No quantity corresponding to leakance is present. To determine the values of these quantities, use is made of the well known equations for a line

$$Z = \sqrt{\frac{R+j\omega L}{G+j\omega C}}; \qquad P = \sqrt{(R+j\omega L)(G+j\omega C)}, \tag{3}$$

where R, L, G and C are respectively the distributed resistance, inductance, leakance, and capacity of the line per unit length. Comparing

[2] Loc. cit.

(3) with (1) and (2), it is found that

$$
\begin{aligned}
R &= \frac{P_0}{S^2}\sqrt{\frac{\gamma'^2 \rho\omega}{2}}, \\
L &= \frac{\rho}{S}, \\
C &= \frac{S}{\rho c^2}, \\
G &= 0,
\end{aligned}
\tag{4}
$$

neglecting small correction terms. These are the equivalent distributed constants per unit length of the pipe expressed in acoustic impedance units.

The representation of lines with distributed constants by means of networks containing lumped constants has received considerable attention.[4] With three impedances, either the T or π network representation shown on Fig. 3, can be used.

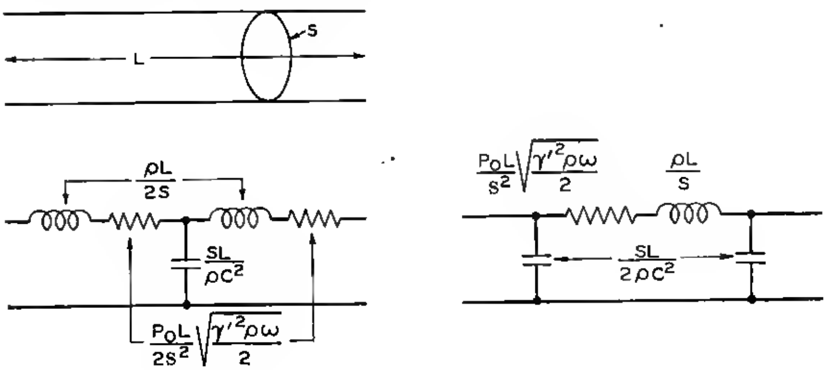


Fig. 3

The impedances of short or open circuited lines can be represented approximately by fewer elements than three. The first approximation for a short circuited line is an inductance and resistance equal to the sum of the distributed inductances and resistances of a line, while the first approximation for an open circuited line will be a capacity equal to the distributed capacities of the line. These approximations hold for very low frequencies. The second approximation for open and short circuited lines can be obtained with three impedances, as shown

[4] A. E. Kennelly "Artificial Electric Lines, 1917."
K. S. Johnson "Transmission Circuits for Telephone Communication, 1925," page 151.

on Fig. 4. These representations follow directly from the T or π

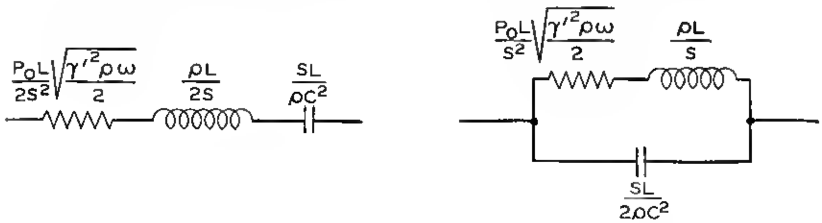


Fig. 4

network representation shown on Fig. 3, by open or short circuiting the T and π networks, respectively.

B. Lumped-Constant Representation of an Acoustic Filter

In his theory of acoustic filters, Stewart has represented an acoustic filter by the network shown on Fig. 5, where Z_2 is the impedance of the

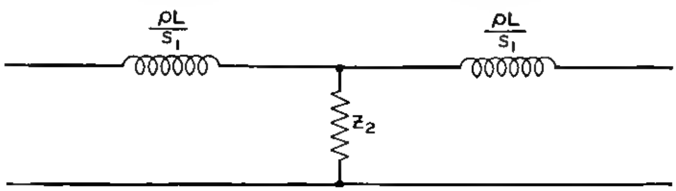


Fig. 5

side branch. Stewart has represented the side branch impedance, by either one or two elements, depending on the side branch, and the main branch by a single inductance, equal to the sum of the distributed inductances of the tube. This corresponds to the first approximation of the representation of a line by lumped constants. This representation gives good results for the low pass filter, but does not represent, very adequately, the band-pass filters.

The best second approximation for an acoustic filter, employing two elements to represent the main conducting tube, is shown on Fig. 6.

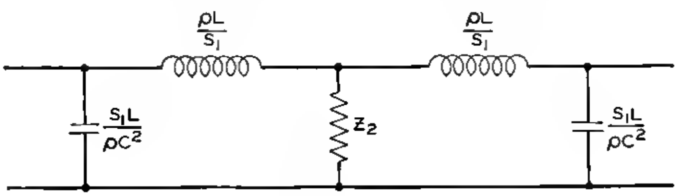


Fig. 6

The main conducting tube is represented by an L network containing the total distributed capacity of the tube in the shunt arm, and the total distributed inductance of the tube in the series arm. The side branch impedance shunts the two L networks at their center.

The propagation constant and characteristic impedance of this structure are given by the expressions

$$\cosh P = 1 - \frac{2\omega^2 L^2}{c^2} + \frac{j\omega\rho L}{Z_2 S_1}\left(1 - \frac{\omega^2 L^2}{c^2}\right),$$

$$Z = \frac{\rho c}{S_1}\sqrt{\frac{1 + \dfrac{j\omega\rho L}{2Z_2 S_1}}{\left[1 - \dfrac{j\rho c^2\left(1 - \dfrac{\omega^2 L^2}{c^2}\right)}{\omega L S_1 (2Z_2)}\right]\left[1 - \dfrac{\omega^2 L^2}{c^2}\right]}}, \tag{5}$$

where S_1 is the area of the main branch.

If these equations are compared with those given in the former paper,[2] it is seen that they are approximately those obtained by taking the first two terms of the expansions of the trigonometrical functions. The characteristics of the filter are not very readily seen from equation (5), but can be readily found by transforming the network shown on Fig. 6, into the much more general lattice network shown in Fig. 7.

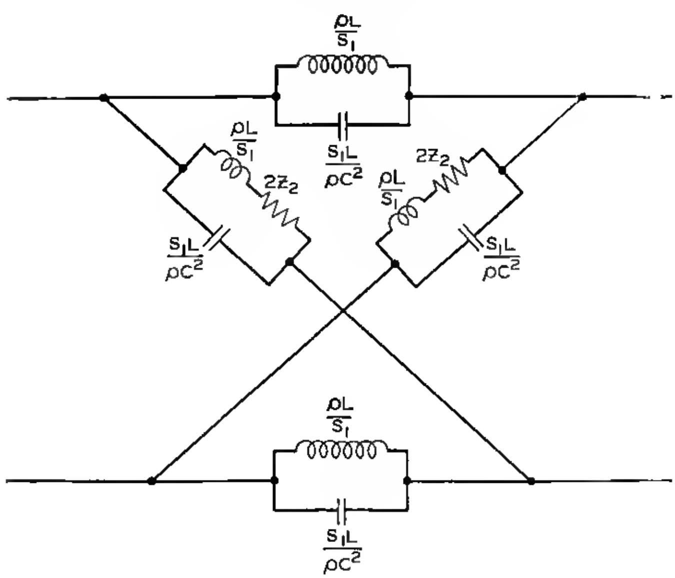


Fig. 7

That the network shown on Fig. 7 is the equivalent in characteristic impedance and propagation constant of that shown on Fig. 6, can readily be verified by substituting the impedances of the lattice network into the formulæ for a lattice network

$$Z = \sqrt{Z_A Z_B}; \qquad \cosh P = \frac{Z_B + Z_A}{Z_B - Z_A}, \tag{6}$$

where Z_A is the impedance of one of the series arms, and Z_B that of one of the lattice arms. A lattice network has a pass band when the reac-

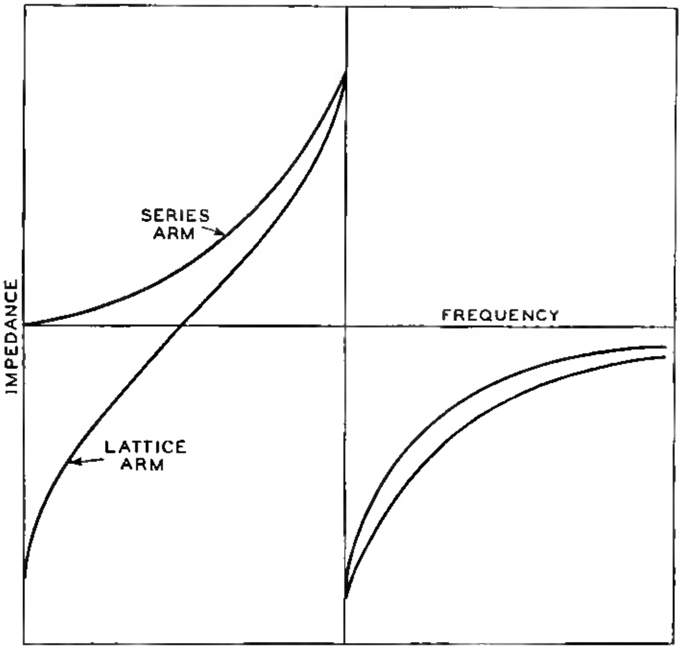


Fig. 8

tance of the series arm is of opposite sign to that of the lattice arm. When the reactances of the two arms have the same sign, an attenuation band results, while when the reactances of the two arms are equal, an infinite attenuation constant results, since here the lattice will be a balanced Wheatstone bridge.

For example, suppose that a side branch impedance, equivalent to an inductance and capacity in series, is used. The impedance of the lattice arm has two zero impedance points—one of which is at an infinite frequency—and two infinite impedance points—one of which is at zero frequency—as shown on Fig. 8. The impedance of the series arm

is that of an anti-resonant circuit, as shown on Fig. 8. There are two possible impedance characteristics for the series arm, in relation to the lattice arm, which will give a single band filter. One of these is obtained by letting the series arm have an infinite impedance when the lattice arm has a zero impedance, which results in a low pass filter. The second relation—which is that shown on Fig. 8—is obtained by letting the series arm have an infinite impedance when the lattice arm has an infinite impedance. The pass band is between zero frequency, and the frequency at which the lattice arm resonates.

In a similar manner, the other types of acoustic filters can be analyzed.

C. Side Branch Impedances

The possible types of side branches can be divided into two classes, those which are entirely enclosed, and those which are open to the air. The first kind are characterized by a series capacity, while the second kind always have a shunt inductance.

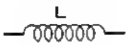
One of the simplest side branch impedances is a short tube open on the end. The first approximation to this side branch is an inductance, as shown on Table I, No. 1, equal to the total distributed inductance of the tube. This approximation holds well if the product of the tube length by the frequency, is not too large. A longer tube, open on the end, can be represented by an inductance and capacity in parallel as discussed in Section A and shown on Table I, No. 2. A tube closed on the end can be represented by an inductance and capacity in series as shown on Table I, No. 4.

When these tubes are used as side branches, an additional factor comes in—an end correction. That is, the side branch must be considered as extending into the main branch for a distance proportional to the radius, because a motion of air in the direction of the side branch, occurs in the main branch. The value of this effect has been investigated by Rayleigh, who found that this effect can be calculated by increasing the length of the tube by a length equal to .785 times the radius. Another correction applies to an open ended tube, which has been determined experimentally as .57 times the radius. Hence the length of an open ended tube must be considered as

$$l' = l + (.785 + .57)r.$$

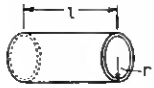
A straight tube can give all the combinations of side branch impedances, but one of its dimensions is necessarily limited, namely the area. For the area cannot become larger than the area of the main tube, since otherwise it could not be connected to the main tube. By

ELEMENT



$$L=\frac{\rho l'}{S}$$

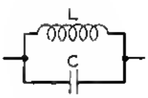
STRUCTURE NO. 1



VALUES OF CONSTANTS

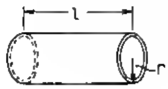
$$l' = l + 1.355r$$
$$S = \pi r^2$$

ELEMENT



$$L=\frac{\rho l'}{S} \qquad C=\frac{l'S}{2\rho c^2}$$

STRUCTURE NO. 2



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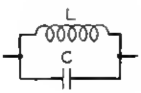
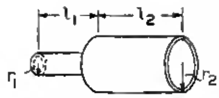


$$L=\frac{\rho l'}{S} \qquad C=\frac{l'S}{2\rho c^2}$$

STRUCTURE NO. 3



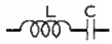
$$l'_1 = l_1+0.785r_1 \qquad S_1=\pi r_1^2$$
$$l'_2 = l_2+0.57r_2 \qquad S_2=\pi r_2^2$$

VALUES OF CONSTANTS

$$l'=\sqrt{\frac{2\left[S_1\left(l_2'^3+3l_1'^2l_2'\right)+\frac{S_1^2}{S_2}\left(3l_1'\,l_2'^2\right)+l_1'^3S_2\right]}{3\left(S_2l_1'+l_2'S_1\right)}}$$

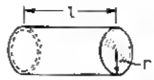
$$S=\sqrt{\frac{2S_1^2S_2\left[S_1S_2\left(l_2'^3+3l_1'^2l_2'\right)+3S_1^2l_1'l_2'^2+S_2^2l_1'^3\right]}{3\left(S_2l_1'+l_2'S_1\right)^3}}$$

ELEMENT



$$L=\frac{\rho l'}{2S} \qquad C=\frac{l'S}{\rho c^2}$$

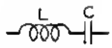
STRUCTURE NO. 4



VALUES OF CONSTANTS

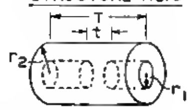
$$l' = l + 0.785r$$
$$S = \pi r^2$$

ELEMENT



$$L=\frac{\rho l'}{2S} \qquad C=\frac{l'S}{\rho c^2}$$

STRUCTURE NO. 5

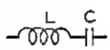


VALUES OF CONSTANTS

$$l'=\sqrt{\left[\frac{\mathrm{Log}\left(\frac{r_1+r_2}{2r_1}\right)}{T}+\frac{0.46}{t}\right]\left[\left(r_2^2-r_1^2\right)T+\frac{r_1^2t}{2}\right]}$$

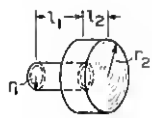
$$S=\pi\sqrt{\frac{\left[\left(r_2^2-r_1^2\right)T+\frac{r_1^2t}{2}\right]}{\frac{\mathrm{Log}\left(\frac{r_1+r_2}{2r_1}\right)}{T}+\frac{0.46}{t}}}$$

ELEMENT



$$L=\frac{\rho l'}{2S} \qquad C=\frac{l'S}{\rho c^2}$$

STRUCTURE NO. 6



$$l'_1 = l_1+0.785r_1 \qquad S_1=\pi r_1^2$$
$$l'_2 = l_2 \qquad S_2=\pi r_2^2$$

VALUES OF CONSTANTS

$$l'=\sqrt{\frac{2\left[\left(l_1'^3+3l_1'\,l_2'^2\right)S_1S_2+3S_1^2l_1'^2l_2'+l_2'^3S_2^2\right]}{3S_2\left[S_1l_1'+S_2l_2'\right]}}$$

$$S=\sqrt{\frac{3S_2\left(S_1l_1'+S_2l_2'\right)^3}{2\left[\left(l_1'^3+3l_1'\,l_2'^2\right)S_1S_2+3S_1^2l_1'^2l_2'+l_2'^3S_2^2\right]}}$$

TABLE I

using other types of side branches, this difficulty can at least be partially eliminated. For example, a concentric tube closed on the end is, to a first approximation, equivalent to an inductance and capacity in series, and it can be made to have a larger area relative to the main branch tube, than can the straight tube.

The choice of the forms of the structures to give the simplest impedance elements, is large. For example, Stewart represents a shunt inductance and capacity in parallel, by a concentric tube closed on the end, and a straight tube open on the end, joined together to the main conducting tube at a common point.[5] Other methods for representing two elements are shown on Table I. In these structures, the equivalent length and equivalent areas have been calculated corresponding to these values for a straight tube. These elements have been calculated by calculating the impedances looking into the structures and taking the second approximations.

D. Design Formulæ for Acoustic Filters

Using the side branch impedances shown in Table I, in the lattice network shown by Fig. 7, the resulting characteristics can readily be obtained. A large number of multiband characteristics can be secured by using various combinations of side branches, but only five single band filters (to the degree of approximation considered here) have been found. The attenuation characteristics of these filters and the design formulæ for them are shown on Table II. In designing a filter, it is usual to obtain the dimensions in terms of the singular frequencies which determine the action of the filter. One other parameter appears, Z_0, which represents the characteristic impedance of the filter at the mean frequency of the band i.e. $f_m = \sqrt{f_1 f_2}$. It is usual to match, approximately, the impedance terminations of the filter to the value Z_0.

All of these filters have been calculated for side branch tubes, of constant cross section but any of the other side branches shown on Table I can be used by employing the equivalent values of l' and S shown there.

The frequency f_a appearing in the filter No. 1 has no significance for the attenuation constant. It determines the frequency at which the characteristic impedance equals infinity. Considering the loss caused by inserting the filter between two impedances equal approximately to Z_0, an additional loss occurs at the frequency f_a, due to a mismatch of the impedance of the filter and the terminating impedances. Filter No. 4 of Table II is similar to No. 3 except that it has twice the attenuation constant. It is then equivalent to two sections of the No. 3 filter.

[5] See for example *Journal of the Optical Society*, July 1929, page 18.

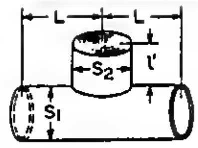
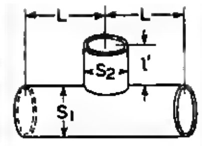
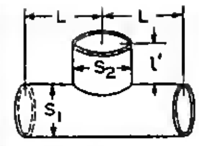
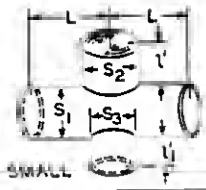
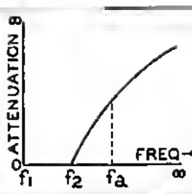
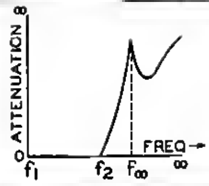
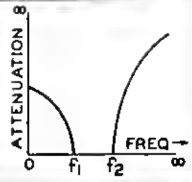
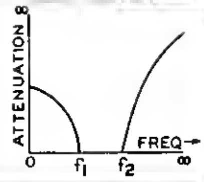
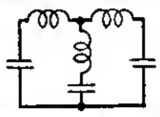
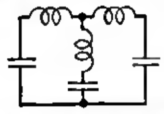
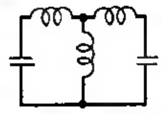
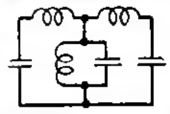
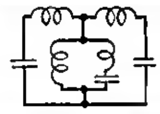
STRUCTURE			 l' IS SMALL		 l_1' IS SMALL
ATTENUATION CHARACTERISTIC					
EQUIVALENT ELECTRICAL STRUCTURE					
VALUES OF: f_1	0	0	$\frac{c}{2\pi L\sqrt{1+\frac{2l'S_1}{LS_2}}}$	$\frac{c}{2\pi L}\sqrt{1-\sqrt{1-\frac{2L^2}{l'^2}}}$	$\frac{c}{2\pi L\sqrt{1+\frac{l_1'S_1}{LS_3}+\frac{l'l_1'S_2}{S_3L^2}}}$
f_2	$\frac{c}{2\pi\sqrt{\left(\frac{L}{S_1}+\frac{l'}{S_2}\right)\frac{l'S_2}{2}}}$	$\frac{c}{2\pi}\sqrt{\frac{1+\frac{l'S_2}{2LS_1}}{\frac{Ll'S_2}{S_1}\left(1+\frac{l'S_1}{LS_2}\right)}}$	$\frac{c}{2\pi L}$	$\frac{c}{2\pi L}\sqrt{1+\sqrt{1-\frac{2L^2}{l'^2}}}$	$\frac{c}{2\pi\sqrt{\frac{l'^2}{2}+\frac{Ll'l_1'}{\frac{LS_3}{S_2}+\frac{l_1'S_1}{S_2}}}}$
f_∞	∞	$\frac{c}{2\pi L}\sqrt{\frac{1+\frac{l'S_2}{2LS_1}}{1+\frac{2l'S_1}{LS_2}-\frac{LS_1}{l'S_2}}}$	∞	∞	$\frac{c}{2\pi L}$ AND ∞
f_a	$\frac{c}{2\pi L}$				
Z_0	$\frac{\rho c}{S_1\sqrt{1+\frac{l'S_2}{2S_1L}}}$	$\frac{\rho c}{S_1}\sqrt{\frac{1}{1+\frac{l'S_2}{2S_1L}}}$	$\frac{\rho c}{S_1}\sqrt{\frac{1+\frac{2l'S_1}{LS_2}}{\left[\sqrt{1+\frac{2l'S_1}{LS_2}}-1\right]^2}}$	$\frac{\rho c}{S_1}\sqrt{\frac{l'^2}{2(l'^2-2L^2)}}$	$\frac{\frac{\rho c}{S_1}\sqrt{1+\frac{2l_1'S_1}{LS_3}}\sqrt{\frac{l'^2}{2L^2}+\frac{\frac{l'l_1'}{L}}{\frac{LS_3}{S_2}+\frac{l_1'S_1}{S_2}}}}{\sqrt{\left(\frac{l'^2}{2L^2}+\frac{\frac{l'l_1'}{L}}{\frac{LS_3}{S_2}+\frac{l_1'S_1}{S_2}}\right)\left(1+\frac{l_1'S_1}{LS_3}+\frac{l'l_1'S_2}{S_3L^2}\right)-1}}$
Z	$\frac{\frac{\rho c}{S_1}\sqrt{\frac{f_\infty^2}{f_2^2}\left(1-\frac{f^2}{f_2^2}\right)}}{1-\frac{f^2}{f_\infty^2}}$	$\frac{\rho c}{S_1}\sqrt{\frac{\frac{f_\infty^2}{f_2^2}\left(2-\frac{f_2^2}{f_\infty^2}-2\sqrt{1-\frac{f_2^2}{f_\infty^2}}\right)}{1-\frac{f^2}{f_2^2}}}$	$\frac{j\rho c}{S_1}\sqrt{\frac{\frac{f^2}{f_1^2}}{\left(1-\frac{f^2}{f_1^2}\right)\left(1-\frac{f^2}{f_2^2}\right)}}$	$\frac{j\rho c}{S_1}\sqrt{\frac{\frac{f^2}{2f_2^2}\left(1+\frac{f_2^2}{f_1^2}\right)}{\left(1-\frac{f^2}{f_1^2}\right)\left(1-\frac{f^2}{f_2^2}\right)}}$	$\frac{j\rho c}{S_1}\sqrt{\frac{\frac{f^2f_2^2}{f_1^2f_\infty^2}\left(1-\frac{f^2}{f_2^2}\right)}{\left(1-\frac{f^2}{f_1^2}\right)\left(1-\frac{f^2}{f_\infty^2}\right)^2}}$
DESIGN FORMULAE FOR: L	$\frac{c}{2\pi f_a}$	$\frac{c}{2\pi f_\infty\left[1-\sqrt{1-\left(\frac{f_2^2}{f_\infty^2}\right)}\right]}$	$\frac{c}{2\pi f_2}$	$\frac{c}{\pi\sqrt{2(f_1^2+f_2^2)}}$	$\frac{c}{2\pi f_\infty}$
S_1	$\frac{f_1\rho c}{Z_0 f_a}$	$\frac{\rho c f_\infty\left[1-\sqrt{1-\left(\frac{f_2^2}{f_\infty^2}\right)}\right]}{Z_0 f_2}$	$\frac{\rho c f_2}{Z_0(f_2-f_1)}$	$\frac{\rho c(f_1^2+f_2^2)}{2Z_0(f_2^2-f_1^2)}$	$\frac{\rho c f_2 f_\infty}{Z_0(f_\infty^2-f_1f_2)}$
l'	$\frac{\sqrt{2}c}{2\pi f_a}$	$\frac{\sqrt{2\left(1-\frac{f_2^2}{4f_\infty^2}-\sqrt{1-\frac{f_2^2}{f_\infty^2}}\right)}}{2\pi f_\infty\left[1-\frac{f_2^2}{2f_\infty^2}-\sqrt{1-\frac{f_2^2}{f_\infty^2}}\right]}$	—	$\frac{c\sqrt{f_1^2+f_2^2}}{2\pi f_1f_2}$	$\frac{c}{\sqrt{2}\pi f_\infty}$
S_2	$\frac{(f_a^2-f_1^2)\rho c\sqrt{2}}{f_1f_aZ_0}$	$\frac{2\left[f_2^2-f_\infty^2\left(1-\sqrt{1-\frac{f_2^2}{f_\infty^2}}\right)\right]}{Z_0f_2\sqrt{2f_\infty^2\left(1-\frac{f_2^2}{4f_\infty^2}-\sqrt{1-\frac{f_2^2}{f_\infty^2}}\right)}}$	—	$\frac{4\rho cf_1f_2(f_1^2+f_2^2)^{\frac{3}{2}}}{Z_0(f_2^2-f_1^2)^2(f_2-f_1)}$	$\frac{\rho c\sqrt{2}\,f_2f_\infty\,(f_\infty^2-f_2^2)}{Z_0(f_\infty^2-f_1f_2)\,(f_2^2-f_1^2)}$
$\frac{l'}{S_2}$	—	—	$\frac{Z_0(f_2^2-f_1^2)(f_1+f_2)}{4\pi\rho f_1^2f_2^2}$	—	—
$\frac{l_1'}{S_3}$	—	—	—	—	$\frac{Z_0(f_2^2-f_1^2)(f_\infty^2-f_1f_2)}{4\pi\rho f_2f_1^2f_\infty^2}$

TABLE II